

Category Theory for Engineers

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1 Introduction

This document was prepared using the hyperref macros in LaTeX web ref. As a consequence, if the reader is connected to the Internet while reading this, when you see an outlined box like the one in the previous sentence or the following one, you can click on that box and hopefully see something interesting like the following NGC 5949. Try it! It is about 44 million light-years away in constellation Draco (one wonders what they are doing over there).

The author of this document was motivated by the NASA document “Engineering Elegant Systems: Principals of System Engineering” by Michael D. Watson EES [8], specifically, postulate two, “the Systems Engineering domain consists of subsystems, their interactions among themselves, and their interactions with the system environment” and postulate seven, “Understanding of the system evolves as the system development or operation progresses”. The aim of the document is to provide a strong foundation for a category theory interpretation of the postulates cited above.

The approach taken here views category theory as an organizational tool for concepts concerned with the design of structures at all levels of size and complexity. Such concepts include physics, mathematics, computational techniques, and the management of group collaborations consisting of possibly diverse subgroups; however these notes will emphasize the mathematical and computational aspects of the applied theory.

It is assumed that the reader is familiar with set theory, statistics, basic algebra and linear algebra, and differential equations at a university graduate level. A good refresher for the necessary set theory is the book [6]. Because of these assumptions, not all of the definitions and concepts in the review of set theory below will be formally complete, but are intended as reminders.

To begin, the important notions of functions, partitions, equivalence relations, and quotient sets will be discussed. These notions are important in ways that will become evident in all that will follow.

The initial discussion leading to the definition of categories will be in terms of directed graphs (digraphs). This approach and some notation involving it has been influenced by the classic reference [7, Chapter I §2]. Examples will be presented both in terms of finite digraphs and much larger mathematical structures where the graph structure is not easy to visualize globally. Specifically, a quick review of set theoretic concepts, an intuitive “categorical view” of sets and functions will be given. Following that, directed graphs (digraphs) and a number of finite digraph based examples will be given and categories will be defined. Then some basic mathematical structures that fit into the categorical framework developed above will be presented. This will include, semigroups, monoids, groups, rings, fields, modules over a ring, vector spaces, and algebras over a “ground ring”. These examples will be used to provide intuition for notions of morphisms which generalize functions, functors which are correspondences between categories that generalize functions, and natural transformations that are correspondences between functors. The notion of natural transformations first appeared in the influential paper [3] which formally introduced category theory.

2 Review of Set Theory

Informally, a set X is a collection of objects which are called elements. If x is an element of a set X we write $x \in X$. If an element in a set is included more than one time, only one copy is considered and the others are ignored. So elements in a set are considered to be distinct. The order in which elements appear in a set is irrelevant. Thus, the set $X = \{2, 7, s, c, c, a, 2\}$ is considered to be the same as $\{7, a, s, c, 2\}$, etc.

A subset of X is a set A such that every element of A is an element of X , i.e. A is “contained in” X . The notation $A \subset X$ is used to denote that A is a subset of X and not all of X itself. To denote that A is a subset that might also be the same as X , the notation $A \subseteq X$ is used.

The *cardinality*, i.e. the number of elements, of a set X is denoted by $\text{card}|X|$ or just $|X|$ when the context is clear. What this means is clear when the number of elements of X is finite. When X is infinite, the situation is more complicated. Intuitively, if the elements in X can be listed in order

indexed by the natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$ we say that X is *countably infinite*. Set theoretically, there are cardinalities larger than “countably infinite” such as the cardinality of the real numbers. The interested reader will find a discussion of infinite cardinal numbers in any standard textbook on set theory.

The union of two subsets A and B of a set X is defined to be

$$A \cup B = \{x \in X \mid x \in A \text{ or } x \in B\}. \quad (1)$$

The intersection of two subsets A and B of a set X is defined to be

$$A \cap B = \{x \in X \mid x \in A \text{ and } x \in B\}. \quad (2)$$

It can occur obviously that A and B have no elements in common. In that case, their intersection is empty and the sets are said to be *disjoint*. The set with no elements is denoted by Φ . So A and B are disjoint if and only if $A \cap B = \Phi$.

The complement of a subset $A \subseteq X$ is

$$A^c = \{x \in X \mid x \notin A\} \quad (3)$$

where the symbol \notin is read “not in”. Another notation that is often used is $A^c = X - A$.

2.1 Functions

A function f from a set X to a set Y is a correspondence such that to each element $x \in X$ there exists a unique element $y \in Y$ that corresponded to it (by assignment). This correspondence is written $y = f(x)$. We also say that $y \in Y$ is “hit by x ” when $y = f(x)$. This is consistent with the convention of calling X the *source* and Y the *target* of f . That convention will be used throughout this note.

Remark 2.1 *Another way of rephrasing the definition above that is useful in defining functions in particular cases of the statement, “if $x = y$ in X , then $f(x) = f(y)$ in Y ”. When a correspondence f satisfies this definition, i.e. is a function, the correspondence is said to be “well-defined”.*

The notation $X \xrightarrow{f} Y$ is quite often used to denote that f a function from X to Y and the assignment of $f(x)$ to x is denoted by $x \mapsto f(x)$ which is read, “ x maps to $f(x)$ ”.

Remark 2.2 A function $X \xrightarrow{f} Y$ is said to be onto if every element in Y is of the form $f(x)$ for some element in X .

The function is said to be one-to-one (or just one-one) if $f(x) = f(x')$ implies $x = x'$.

Remark 2.3 An important “operation” involving functions is called composition. Given two functions $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ there is an associated function $X \xrightarrow{g \circ f} Z$ defined by $(g \circ f)(x) = g(f(x))$ for each x in X .

In addition, as pointed out in section 2.5, for any set X , there is the identity function $X \xrightarrow{id_X} X$ defined by $id_X(x) = x$ for all $x \in X$. It is noted here that with respect to composition of functions, for an function $X \xrightarrow{f} X$ we have that

$$(id_X \circ f)(x) = id_X(f(x)) = f(x), \quad \text{and} \quad (4)$$

$$(f \circ id_X)(x) = f(x) \quad (5)$$

so that we always have $id_X \circ f = f$ and $f \circ id_X = f$ for functions f that map X to X .

2.2 Inverse Image and Partitions

Given a function $X \xrightarrow{f} Y$ and $y \in Y$, the *inverse image* of y is the subset

$$f^{-1}(y) = \{x \in X \mid f(x) = y\}. \quad (6)$$

Consider the set

$$P = \{f^{-1}(y) \mid y \in Y\}. \quad (7)$$

Remark 2.4 If $f^{-1}(y) \cap f^{-1}(y') \neq \Phi$, then there is an element $x \in X$ such that $f(x) = y$ and $f(x) = y'$ and since f is well-defined (see remark (2.1)), this implies that $y = y'$. This means that the distinct elements of the set of inverse images P above are all pairwise disjoint.

Also note that the union of all of the sets in P is all of X since the union is clearly a subset of X , but if $x \in X$, then obviously, $x \in f^{-1}(f(x))$ so that x is in the union of all inverse images.

A set of pairwise disjoint subsets of a set X whose union is all of X is called a partition of X .

2.3 Products of Sets

If X_1, X_2, \dots, X_n is a list of sets for any $n \geq 2$ the product of these n sets in the given order is

$$X_1 \times X_2 \times \dots \times X_n = \{(x_1, x_2, \dots, x_n) \mid x_i \in X_i, i = 1, \dots, n\}. \quad (8)$$

The element (x_1, x_2, \dots, x_n) is called an n -tuple. When $n = 2$ it is usually called an “ordered pair”.

2.4 Relations

A relation in X is any subset $R \subseteq X \times X$. The notation $x \xrightarrow[R]{} x'$ if and only if $(x, x') \in R$ will often be used in this note.

Remark 2.5 *The relation*

$$\Delta = \{(x, x) \in X \times X \mid x \in X\} \quad (9)$$

is the relation of equality. A given element $x \in X$ is related to an element x' if and only if $x = x'$. The equality relation Δ is also called the diagonal.

Note that the graph $G_f = \{(x, f(x)) \mid x \in X\}$ of a function f is a relation in X . So functions may be thought of as special kinds of relations. In fact, the graph of the *identity* function

$$X \xrightarrow{id_X} Y \quad (10)$$

$$x \mapsto x \quad (11)$$

is Δ which is the geometric reason it is called the diagonal.

Remark 2.6 *Given a relation $R \subseteq X \times X$ the opposite relation is*

$$R^{op} = \{(x', x) \mid (x, x') \in R\}. \quad (12)$$

2.5 Equivalence Relations and Quotient Sets

A relation $E \subseteq X \times X$ is called an *equivalence relation* if and only if it satisfies the three properties given by

1. $\Delta \subseteq E$, i.e. $x \xrightarrow{E}$ for all $x \in X$.
2. $E^{op} \subseteq E$, i.e. if $x \xrightarrow{E} x'$ then $x' \xrightarrow{E} x$, and
3. if $x \xrightarrow{E} x'$ and $x \xrightarrow{E} x''$ then $x \xrightarrow{E} x''$

A relation E that satisfies 1. above is called *reflexive*. It is called *symmetric* if it satisfies 2. above and it is called *transitive* if it satisfies 3..

An equivalence class for an equivalence relation E on X is defined for each element of X to be

$$[x]_E = \{x' \in X \mid x \xrightarrow{E} x'\}. \quad (13)$$

When the context is clear, the subscript E is dropped from $[x]_E$ and we simply write $[x]$ for the equivalence class of x .

Remark 2.7 *The set of equivalence classes in X is defined to be the set of subsets*

$$X/E = \{[x] \mid x \in X\}. \quad (14)$$

In fact, X/E is a partition of X . To see why, assume that $[x] \cap [x] \neq \Phi$. Then there is some element z in both classes, i.e. $x \xrightarrow{E} z$ and $x' \xrightarrow{E} z$ so since $z \xrightarrow{E} x$ by symmetry and transitivity, we have $x' \xrightarrow{E} x$. Similarly, $x \xrightarrow{E} x'$.

So suppose that $x'' \in [x]$. Then symmetry and transitivity with the above implies $x'' \xrightarrow{E} x'$ so that $[x] \subseteq [x']$. Analogously, $[x'] \subseteq [x]$ since every element x'' will also be in $[x]$ by the same reasoning and so $[x] = [x']$ and hence X/E is a partition.

The set X/E is called the quotient set of X by E . It is an important construction in mathematics. Note that if x and x' are related in X by E , then $[x] = [x']$ in X/E . Thus intuitively, taking quotient sets turns “equivalence” to “equality”.

Remark 2.8 While it is true that equivalence “becomes” equality at the level of set theory, one has to be careful not to overgeneralize such a notion to other situations that will be encountered later in category theory.

Exercise 2.1 Consider an equivalence relation E on a set X and the correspondence defined by

$$X \xrightarrow{q} X/E \quad (15)$$

$$x \mapsto [x]. \quad (16)$$

1. Prove that q is well-defined, i.e. if $x = x'$ then $q(x) = q(x')$.
2. How does the partition $P = \{q^{-1}([x]) \mid [x] \in X/E\}$ compare to the partition X/E ?
3. What does this say about functions, equivalence relations, and partitions?

Exercise 2.2 Let $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ denote that set of all integers.

Recall the result of the division algorithm from grammar school (sometimes called “long division”):

Given an integer $a \in \mathbb{Z}$ and a non-negative integer n , there is an integer $q \in \mathbb{Z}$ (called the quotient) such that $a = qn + r$ where $r \in \mathbb{Z}$ and $0 \leq r < n$.

Fix a non-negative integer n .

1. Prove that the relation mod_n

$$a \xrightarrow[\text{mod}_n]{} b \text{ if and only if } a = b + qn \text{ for some } q \in \mathbb{Z} \quad (17)$$

defines an equivalence relation on \mathbb{Z} .

2. Prove that for each $[a] \in \mathbb{Z}/\text{mod}_n$ there is a unique non-negative integer r such that $[a] = [r]$ so that the set of equivalence classes \mathbb{Z}/mod_n is in one-to-one correspondence with the finite set $\{0, 1, \dots, n-1\}$.

Note that we usually simply write \mathbb{Z}/mod_n as \mathbb{Z}/n

Exercise 2.3 *Show that the intersection of any number of equivalence relations is an equivalence relation. Thus if $R \subseteq X \times X$ is any relation, there is a smallest equivalence relation containing it, viz, the intersection of all equivalence relations containing it (note that $X \times X$ is an equivalence relation that contains any such R)*

Remark 2.9 *Note however that your proof may not be constructive. There are algorithms however that do construct such a smallest equivalence relation “extension” of a given R . The easy part is making a given R reflexive and symmetric in a minimal way. If it is not reflexive, simply take $R' = \Delta \cup R$ where Δ is the diagonal. If that is not symmetric, take $R'' = R' \cup (R')^{\text{op}}$. R'' will then contain R and be reflexive and symmetric. The “tedious” part is in making R'' transitive if it is not already so. Doing that in a minimal way is said to produce the transitive closure of R'' which will then be an equivalence relation. There are efficient algorithms for calculating the transitive closure. A search on the Internet for “transitive closure” recently produced 852,000 results. Of note is Warshall’s algorithm and more recently parallelized versions which may also be found on the Internet.*

Remark 2.10 *Note that since the entire product set $X \times X$ is an equivalence relation, any relation $R \subseteq X \times X$ is contained in a unique equivalence relation, namely the intersection of all equivalence relations in X that contain R .*

For an interesting discussion of quotient sets in topology, the interested reader should see [2, §4.5 Adjunction spaces].

3 Some Observations Concerning Sets

There are logical problems in thinking about the notion of a set of all sets. One cannot simply allow any proposition to define a set. The classic example is Russell’s paradox [5] which is about the specification of a set X which does not contain itself as an element (one asks if X is an element of X and considers the consequences. Then one asks if X is not an element of X to see the paradox). For many reasons including such paradoxes (there are more than just Russell’s), there are axiomatic treatments of set theory [11] which avoid such paradoxes. There are various conventions for talking about a container for all sets and other collections that are “too big” to be

sets. A discussion of such conventions is given in [7, Chapter I, §6,§7.]. The convention taken here is to talk about a kind of universal container, called a *class* which may contain all sets without logical difficulties. The formal reasons that this can be done require a proper reading of references such as the three mentioned above and will be left to the interested reader.

A class that is not a set will be called a *proper class*. A class that is not a proper class is called a *small* class.

3.1 Sets as Nodes, Functions as Arrows

A notation for specifying a function f from a set X to Y , viz. $X \xrightarrow{f} Y$ has already been discussed. It suggests a kind of graph structure on the class of all sets. Indeed, in section 2.1, X has already been called the source and Y the target of f , so we alternately call a function f an arrow in this context. This leads to a view of sets and functions as “nodes” and “arrows” of a graph structure (a rather large one admittedly).

3.2 The Set of All Arrows from one Set to Another

We will often denote the set of all functions from a set X to a set Y by $[X, Y]_{Set}$. Here we will consider the number of possible functions from one finite set to another, i.e. the cardinality of the set $[X, Y]_{Set}$ when X and Y are finite. To begin, consider the set $X = \{0\}$ with one element and the set $Y = \{0, 1, \dots, n-1\}$ with n elements. Clearly, the number of choices of a correspondence of 0 to a unique element in Y is exactly n . We may denote these functions by $f_i(0) = i$ for $i = 0, \dots, n-1$.

Now consider all functions from $X = \{0, 1\}$ to itself. All such functions may be conveniently denoted by

$$f_{i,j} = \begin{pmatrix} 0 & 1 \\ i & j \end{pmatrix}. \quad (18)$$

where $i, j \in Y$ with the possibility that $i = j$. It is easy to write down all of the choices, viz.

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}. \quad (19)$$

Continuing, the distinct functions from X to $Y = \{0, \dots, n-1\}$ may be enumerated by filling in the symbols i, j in the expression in (18) above

where this time $i, j \text{ in } Y = \{0, \dots, n-1\}$ (with the possibility that $i = j$ again. Obviously, there are n choices for associating (or “mapping”) 0 and following that, there are n choices for mapping 1. In all, this makes $n \cdot n = n^2$ choices.

Finally, consider the number of functions from $X = \{0, \dots, n-1\}$ to $Y = \{0, \dots, m-1\}$ which can be enumerated by the expressions

$$\begin{pmatrix} 0 & \dots & n-1 \\ i_0 & \dots & i_{n-1} \end{pmatrix} \quad (20)$$

where $i_j \in Y$ with possible repeats. Again, clearly, there are m choices for 0, m choices for 1 and so on, making $m \cdot m \dots m = m^n$ choices in all.

Remark 3.1 *A moment of thought shows that the same argument can be made for any finite sets X and Y in terms of counting functions because the counting argument does not depend upon what the elements of a finite set are labeled. The counting is the same is we label elements as 0, 1, 2 or a, b, c , etc.*

Thus, we see that the cardinality of $[\{0, \dots, n-1\}, \{0, \dots, m-1\}]_{\text{Set}}$ is exactly m^n . In other words,

$$|[X, Y]_{\text{Set}}| = |Y|^{|X|}. \quad (21)$$

Because of this result, the notation Y^X is sometimes used to denote $[X, Y]_{\text{Set}}$.

3.3 A Small Piece of the Set Theory Graph

Consider just the collection of sets $\{0\}$, $\{0, 1\}$ and their arrows. Let $\bar{0} = \{0\}$ and $\bar{1} = \{0, 1\}$. There $1^1 = 1$ function from $\bar{0}$ to itself, $2^1 = 2$ functions from $\bar{0}$ to $\bar{1}$, $1^2 = 1$ function from $\bar{1}$ to $\bar{0}$, and $2^2 = 4$ functions from $\bar{1}$ to itself. Thus, there are 8 arrows involving the two sets $\bar{0}$ and $\bar{1}$.

Exercise 3.1

- Draw a graph with two nodes labeled $\bar{0}$ and $\bar{1}$ and the 8 arrows mentioned above. Note that 3 of the arrows map one node to another while one arrow is a “loop” at one node and 4 arrows are loops at the other node.
- Construct a table that identifies all possible compositions (as functions) or the 8 arrows for which it is possible to form composite functions.

Remark 3.2 *Note that the exercise above hints at how complicated the full graph of all finite sets of the form $\{0, 1, 2, \dots, n-1\}$ is for $n \leq 1, 2, 3, \dots$. Adding the thrust of remark (3.1) to this observation, one can see a kind of crystalline structure in the “set theory graph”.*

4 Directed Graphs and Free Path Algebras

A directed graph (digraph) $\mathcal{G} = \{G_0, G_1; s, t\}$ consists of two sets G_0 and G_1 along with two functions $G_1 \xrightarrow{s} G_0$, $G_1 \xrightarrow{t} G_0$. The function s is called the *source* function and t is called the *target* function. We will use the following notation for this situation.

$$G_1 \xrightleftharpoons[t]{s} G_0, \quad (22)$$

The elements of G_0 are called *nodes* or *vertices*. The elements of G_1 are called *arrows*.

An arrow $a \in G_1$ such that $s(a) = t(a)$ is called a loop arrow or just a *loop* (at the node $s(a)$).

The following implicitly defines a typical digraph.

$$G_0 = \{a, b, c, d, e\} \quad (23)$$

$$G_1 = \{[a, b], [b, e], [c, c], [c, d], [d, c], [e, e], [e, d], [e, a]\}. \quad (24)$$

The arrows are denoted by lists of the form $\alpha = [x, y]$ where $s(\alpha) = x$ and $t(\alpha) = y$. This is not a universally used convention, but in some instances, it is convenient. Thus, the loops in the above graph are $[c, c]$ and $[e, e]$. Figure (1) illustrates the corresponding digraph.

4.1 The Free Non-Associative Path Algebra

The reason for the title word “Non-Associative” will be explained in the next section.

Given a digraph $\mathcal{G} = \{G_0, G_1; s, t\}$, define a sequence of sets $\widehat{G}_{1,n}$ inductively as follows.

- $\widehat{G}_{1,1} = G_1$.

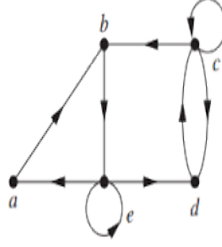


Figure 1: A Typical Digraph

- $\widehat{G}_{1,2}$ is the set of all parenthesized juxtapositions (ab) of elements in $\widehat{G}_{1,1}$ such that $s(b) = t(a)$, i.e. $\widehat{G}_{1,2} = \{(ab) \mid a, b \in \widehat{G}_{1,1}, s(b) = t(a)\}$.
- For $(ab) \in \widehat{G}_{1,2}$, define an extension of the source and target functions by $s((ab)) = s(a)$ and $t((ab)) = t(b)$.

Suppose that we have $G_{1,k}$ as well as extensions of s and t for $2 \leq k < n$

- $\widehat{G}_{1,n}$ is the set of all parenthesized juxtapositions of elements (cd) where $c \in G_{1,i}$ and $d \in G_{1,j}$ with $1 \leq i, j$ and $i + j = n$ and $s(d) = t(c)$.

Note that every element $\alpha \in \widehat{G}_{1,n}$ consists of a sequence $a_1, a_2, \dots, a_n \in G_1$ with parentheses in various positions between the a_i s and $s(\alpha) = s(a_1), t(\alpha) = t(a_n)$.

Now define

$$\widehat{G}_1 = \cup_{n \geq 1} \widehat{G}_{1,n} \quad (25)$$

and let

$$\widehat{G}_1 \times_{G_0} \widehat{G}_1 = \{(a, b) \in \widehat{G}_1 \times \widehat{G}_1 \mid s(b) = t(a)\}. \quad (26)$$

Finally, define an operation called “composition” by

$$\widehat{G}_1 \times_{G_0} \widehat{G}_1 \rightarrow \widehat{G}_1 \quad (27)$$

$$(a, b) \mapsto (ab). \quad (28)$$

This operation is often denoted by $a \circ b = (ab)$.

Note that the operation is well defined since any element a must be an element of some $\widehat{G}_{1,i}$ and similarly, b must be an element of some $\widehat{G}_{1,j}$ so $(ab) \in \widehat{G}_{1,i+j} \subseteq \widehat{G}_1$.

4.2 Degree of a Word

As defined above, every element $\alpha \in \widehat{G_1}$ consists of a sequence $a_1, \dots, a_n \in G_1$ with parentheses in various positions between the a_i s for some n . We call α a *word* in the indicated elements. We define the *degree* of such an element to be $\deg(\alpha) = n$. More can be said however, by construction, α is actually a concatenation of an element from $\widehat{G_1, i}$ and $\widehat{G_1, j}$ where $i + j = n$. We define the *bidegree* of α to be (i, j) .

4.3 The Free Path Algebra

Associativity Generally, a function $S \times S \xrightarrow{m} S$ such that $m(m(a, b), c) = m(a, m(b, c))$ for all $a, b, c \in S$ is called an *associative* operation. If we write $m(a, b) = ab$, this becomes the more familiar rule that $a(bc) = (ab)c$ as is known to hold for the usual operation of multiplications of integers and composition of functions, for example. Because these two expressions are equal, one can unambiguously write abc for either one of them thereby forgetting about any parentheses in products of such elements.

The free associative path algebra on graph $\mathcal{G} = \{G_0, G_1; s, t\}$ is a mathematical structure that is much easier to visualize than the free non-associative one because of the comments above. In $\widehat{G_2}$ there are distinct elements of the form $(a(bc))$ and $(a(bc))$ which indeed indicates that the path product m is not associative in general, but if associativity is assumed, all parentheses can be removed and all words of degree n are simply concatenations of n elements from G_1 such that the source of a factor of the concatenation whose position is greater than one is the target of the previous factor.

Generally, we call the free associative path algebra simply the free path algebra, dropping the word “associative”. If we need to refer to the non-associative path algebra, the word “non-associative” will be explicitly used.

The free path algebra on a graph $\mathcal{G} = \{G_0, G_1; s, t\}$ will be denoted by $\mathcal{P}(\mathcal{G})$.

Example 4.1 Consider the digraph defined by $G_0 = \{1\}$, $G_1 = \{a\}$ and $s(a) = 1$; $t(a) = 1$. This graph is illustrated in figure (2). Clearly, in this case, the set of all path products is $\widehat{G_1} = \{a, aa, aaa, aaaa, \dots\}$. We can abbreviate the notation by writing $a^1 = a$ and $aa \dots a$ (n -times) as a^n . By the definition of path product and associativity, we then clearly have $a^n \circ a^m = a^{n+m}$.



Figure 2: A one node, one arrow digraph

Remark 4.1 *A set S with an operation*

$$S \times S \xrightarrow{\circ} S \quad (29)$$

which is associative is called a semigroup. Thus, in example above, the free path algebra \widehat{G}_1 is an example of a semigroup.

4.4 Identities

In ordinary multiplication of integers which we write as \mathbb{Z} , the distinguished element 1 has the property that $1 \cdot n = n$ and $n \cdot 1 = n$ for all $n \in \mathbb{Z}$. Such an element is called an *identity element*. The path algebra on a graph \mathcal{G} may possess identity elements “at each node” if there is a distinguished arrows that satisfy the property that, in addition to the source and target maps s and t , there is a function

$$G_0 \xrightarrow{ids} G_1 \quad (30)$$

such that

$$s \circ ids = id_{G_0}, \quad \text{and} \quad t \circ ids = id_{G_0} \quad (31)$$

These conditions imply that at each node $n \in G_0$, there is an arrow which we will denote by $id_n = ids(n)$ such that $s(id_n) = s(ids(n)) = n$ and $t(id_n) = t(ids(n)) = n$, i.e. that each id_n is a loop at n for all nodes in G_0 .

With the conditions above, it is assumed that the id_n loops act as identity elements in the path algebra $\mathcal{P}(\mathcal{G})$. This combined structure is described succinctly below.

4.5 The Free Path Algebra With Identities

The free path algebra with identities has following structure.

$$\mathcal{P}(\mathcal{G}) \begin{smallmatrix} \xrightarrow{s} \\ \xrightarrow{t} \end{smallmatrix} G_0, \quad (32)$$

along with

$$G_0 \xrightarrow{id_s} \mathcal{P}(\mathcal{G}) \quad (33)$$

such that

$$s \circ id_s = id_{G_0} \quad (34)$$

$$t \circ id_s = id_{G_0} \quad (35)$$

with an induced associative operation

$$\widehat{G}_1 \times_{G_0} \widehat{G}_1 \xrightarrow{m} \widehat{G}_1 \quad (36)$$

such that the loops id_n for $n \in G_0$ act as identities.

Example 4.2 Consider the digraph defined by $G_0 = \{0\}$, $G_1 = \{id_0, 1\}$, $s(1) = 0$; $t(1) = 0$, $s(id_0) = 0$; $t(id_0) = 0$ and identity $ids(0) = id_0$. This graph is illustrated in figure (3) (with the identity arrow id_0 omitted but assumed). (2). In this case, we will write the operation of path product as $+$.



Figure 3: A one node, one arrow, one identity digraph

Thus, the set of all path products is $\widehat{G}_1 = \{id_0, 1, 1 + 1, 1 + 1 + 1, \dots\}$. Again we can abbreviate the notation, but this time, we will write $1 + 1 + \dots + 1$ (n -times) as n . By the definition of path product and associativity, we then clearly have $n \circ m = n + m$ where, on the left we mean $1 + \dots + 1$ $n + m$ -times. Note that in this case, we also have $n + m = m + n$ as is clear from the definition of path product in this case. Also, by the properties of identity loops, we have $id_0 + n = n$ and $n + id_0 = n$. We denote id_0 in this case by 0 .

Remark 4.2 *A set M with an operation*

$$M \times M \xrightarrow{\circ} M \quad (37)$$

which is associative and has an identity element is called a monoid. Thus, in example above, the free path algebra \widehat{G}_1 with identities is an example of a monoid (where we write the operation as $+$ instead of \circ in this case, but more can be said. We have that

$$\widehat{G}_1 = \{0, 1, 2, \dots, n, \dots\} \quad (38)$$

with $0 + n = n$, and $n + m = m + n$. This monoid is the same as the set of non-negative integers with addition and identity element zero.

5 Categories

The above free (associative) path algebra with identities is a model for the general definition of a category; however, while this mathematical structure was *constructed* artificially, we have seen a naturally occurring example, viz. sets, functions, composition of functions, and identity maps. This last comment needs some explanation and that is given below where we present sets and functions as a category.

In general, while we take the free path algebra with identities as a *model* for categories, we do not require a category to be exactly of this form, In fact, the definition of a category is as follows.

Definition 5.1 *A category \underline{C} consists of two classes C_0 and C_1 and two well defined correspondences*

$$C_1 \xrightleftharpoons[t]{s} C_0 \quad (39)$$

so that $s \circ ids = id_{C_0}$ and $t \circ ids = id_{C_0}$. It furthermore is supposed that there is an operation \circ of the form

$$C_1 \times_{C_0} C_1 \xrightarrow{\circ} C_1 \quad (40)$$

(where $C_1 \times_{C_0} C_1 = \{(f, g) \mid C_1 \times C_1, s(g) = t(f)\}$) which is associative and for which there are identities with respect to this operation.

$$C_0 \xrightarrow{ids} C_1 \quad (41)$$

We will always denote categories with an underline as in \underline{C} .

Remark 5.1 *The free path algebra with identities satisfies this definition with respect to \circ given by the path product. Thus, $\underline{G} = (\widehat{G}_1, G_0, s, t, ids)$ is a category. It is called the free category on the digraph with identity loops (G_0, G_1, s, t, ids) .*

Example 5.1 *Here is an Interesting yet very simple category.*

Consider the digraph given by $G_0 = \{n\}$, $G_1 = \{id_n, 0, 1\}$, $s(0) = n; t(0) = n$, $s(1) = n; t(1) = n$, and identity $ids(n) = id_n$. This graph is illustrated in figure (4) (again with the identity loop omitted but assumed).

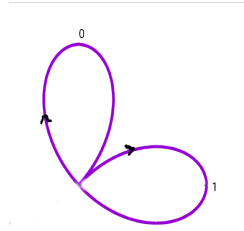


Figure 4: One node, two arrows, and one identity arrow

This time, \widehat{G}_1 consists of the identity loop id_n and all words in 0 and 1. Note however that 01 is not the same as 10. Nonetheless, this set is a monoid.

Example 5.2 *Consider the digraph with \underline{Set}_0 equal to the proper class of all sets and \underline{Set}_1 equal to the class of all functions from a set to any other set. For a function $X \xrightarrow{f} Y$, we have $s(f) = X$ and $t(f) = Y$. We interpret the path product as composition of functions. The identities are $ids(X) = id_X$. Since composition of functions is associative, this structure which will be denoted by \underline{Set} is a category.*

Remark 5.2 *Consider again the category from example (5.1) which we will denote by \underline{Mach} . If we interpret the words not equal to the identity id_n as encoding words in English via the ASCII encoding [1], the phrase*

01001000 01100101 01101100 01101100 01101111
01110111 01100111 01110010 01101100 01100100

which reads “Hello world” is encoded in the category Mach.

It is interesting to note that all of written history may be encoded in this small category.

5.1 Some Standard Terminology

Let $\underline{C} = (\underline{C}_0, \underline{C}_1, s, t, ids)$ be a category. The class of nodes \underline{C}_0 is often called $Ob(\underline{C})$ and the class of arrows \underline{C}_1 is called $Arr(\underline{C})$. For two objects (nodes) $c_1, c_2 \in Ob(\underline{C})$ the class of all arrows from c_1 to c_2 is denoted by either $[c_1, c_2]_{\underline{C}}$ or $hom_{\underline{C}}(c_1, c_2)$.

5.2 A Strong Correspondence Between Two Categories

Consider two one node two loop categories generated freely by the graphs in figure (5). Let \underline{C} denote the category generated by the digraph on the left.

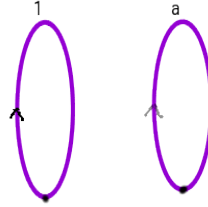


Figure 5: Two one node, one arrow, one identity digraphs

As in example 4.2, let n denote $1 + \dots + 1$ (n -times) and let 0 denote id_0 .

The set of arrows of \underline{C} is

$$Arr(\underline{C}) = \{id_0, 1, 2, \dots, n, \dots, \} \quad (42)$$

and the path product is $n \circ m = n + m$.

As in example 4.1, let a^n denote $aa \dots a$ (n -times). For this example however, we include the identity id_1 which we denote by 1 . The set of arrows is

$$Arr(\underline{C}') = \{1, a, aa, \dots, a^n, \dots\} \quad (43)$$

and the path product is $a^n \circ a^m = a^{n+m}$.

Now it is quite apparent that these two categories are *essentially* the same. However, the word “essentially” needs to be made more

To make a complete comparison, we need to compare nodes, identity loops, and other arrows of both categories. That means that we need not only a correspondence between $Ob(\underline{C})$ and $Ob(\underline{C}')$, but also one between $Arr(\underline{C})$ and $Arr(\underline{C}')$.

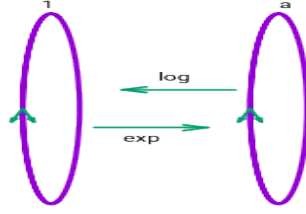


Figure 6: Comparing two categories.

Figure (6) gives such correspondences, viz. we define

$$Ob(\underline{C}) \xrightarrow{exp} Ob(\underline{C}') \quad (44)$$

$$0 \mapsto 1 \quad (45)$$

and

$$Arr(\underline{C}) \xrightarrow{exp} Arr(\underline{C}') \quad (46)$$

$$n \mapsto a^n. \quad (47)$$

Both of these correspondences are well-defined and furthermore, they are clearly one-one and onto and have inverses. In fact, we have that the inverse functions are given by

$$Ob(\underline{C}') \xrightarrow{log} Ob(\underline{C}) \quad (48)$$

$$1 \mapsto 0 \quad (49)$$

and

$$Arr(\underline{C}') \xrightarrow{log} Arr(\underline{C}) \quad (50)$$

$$a^n \mapsto n. \quad (51)$$

Note furthermore, that these functions preserve path products, viz. $\exp(n+m) = a^{n+m} = a^n a^m = \exp(n)\exp(m)$ and $\log(a^n a^m) = \log(a^{n+m}) = n+m = \log(a^n) + \log(a^m)$. Because of this, we say that the two categories are isomorphic, i.e. they are essentially the same category.

5.3 Functors

A functor between two categories is a generalization of the above example; however, in general, they are much weaker than being isomorphisms, of course.

A functor $\underline{C} \xrightarrow{f} \underline{D}$ consists of correspondences of the form

$$Ob(\underline{C}) \xrightarrow{f} Ob(\underline{D}) \quad (52)$$

and

$$hom_{\underline{C}}(C, D) \xrightarrow{f} hom_{\underline{D}}(f(C), f(D)) \quad (53)$$

for every pair of objects $C, D \in \underline{C}$ such that compositions are preserved, i.e. on morphisms, $f(\beta \circ \alpha) = f(\beta) \circ f(\alpha)$.

Such a functor is sometimes called a *covariant* functor because it preserves the *direction* of arrows. Shortly, we will see an example of a functor that reverses the direction of arrows. That kind of functor is called a *contravariant* functor.

5.4 The Dual of a Category

Given a category $\underline{C} = (\underline{C}_0, \underline{C}_1, s, t, ids)$, the dual category \underline{C}^{op} has the same class of objects as \underline{C} , but its arrows are reversed, i.e. $s^{op} = t$ and $t^{op} = s$.

Note that a contravariant functor $\underline{C} \xrightarrow{f} \underline{D}$ is the same as a covariant functor $\underline{C}^{op} \xrightarrow{f} \underline{D}$

5.5 Relations in a Category

Relations in a category \underline{C} are relations R_{c_1, c_2} in $hom_{\underline{C}}(c_1, c_2)$ for all $c_1, c_2 \in Ob(\underline{C})$. Thus, $R_{c_1, c_2} \subseteq hom_{\underline{C}}(c_1, c_2) \times hom_{\underline{C}}(c_1, c_2)$. As with sets, a relation

E_{c_1, c_2} is called an equivalence relation if it is reflexive, symmetric, and transitive. We say that the family $\{E_{c_1, c_2}\}$ is preserved by compositions if the correspondence

$$\text{hom}_{\underline{C}}(c_1, c_2)/E_{c_1, c_2} \times \text{hom}_{\underline{C}}(c_2, c_3)/E_{c_2, c_3} \rightarrow \text{hom}_{\underline{C}}(c_1, c_3)/E_{c_1, c_3} \quad (54)$$

given by

$$[g] \circ [f] = [g \circ f] \quad (55)$$

is well-defined (see section 2.1).

Exercise 5.1 *Being well-defined in the case directly above means that the correspondence*

$$([g], [f]) \mapsto [g \circ f] \quad (56)$$

actually defines a functional correspondence. In other words, the assignment is unique, and in still other words, that if $([g], [f]) = ([h], [k])$ then $[g \circ f] = [h \circ k]$.

The above amounts to saying that if g is related to g' and h is related to h' , then $g \circ f$ is related to $h \circ k$. So there is nothing really to “prove” here, it is an exercise in unraveling definitions. The exercise, such as it is, is to identify precisely where all of this is “going on” in the category in question. That involved the specific “hom”s and $E_{c, c'}$ s involved, etc.

Remark 5.3 *If it is required to design a category for some specific use one way to proceed consists of specifying an appropriate digraph, specifying relations in the corresponding free category that the digraph generates and extending those relations to equivalence relations that preserve compositions (as specified in the exercise above). The category formed by taking the quotient classes of the arrows by the given equivalence relations is then the desired end category.*

It is possible to extend relations as above to equivalence relations that preserves compositions. Earlier, an exercise (2.3) showed that there is a smallest equivalence relations containing a given relation. The same consideration may be used to construct a smallest equivalence relation that preserves compositions (when they are present). The interested reader should also see [7, Chapter I §8].

5.6 Some Basic Mathematical Categories

5.7 The Category of Semigroups

Semigroups were defined in section 4.1. The category \underline{SemiGp} has objects consisting of all semigroups and arrows (or morphisms as they are also called) all functions $S_1 \xrightarrow{f} S_2$ on the underlying sets that satisfy $f(st) = f(s)f(t)$. The set of arrows is denoted as usual by either $[S_1, S_2]_{\underline{SemiGp}}$ or $hom_{\underline{SemiGp}}(S_1, S_2)$. Such morphisms are called semigroup maps or semigroup morphisms.

5.8 The Category of Monoids

Monoids were defined in section 4.2. The category \underline{Monoid} has object consisting of all monoids and arrows (or morphisms as they are also called) all functions $M_1 \xrightarrow{f} M_2$ on the underlying sets that satisfy $f(st) = f(s)f(t)$ and $f(1) = 1$. The set of arrows is denoted as usual by either $[M_1, M_2]_{\underline{Monoid}}$ or $hom_{\underline{Monoid}}(M_1, M_2)$. Such morphisms are called monoid maps or monoid morphisms.

5.9 The Category of Groups

A group G is a monoid such that every element $g \in G$ has an inverse denoted by g^{-1} . That means that $gg^{-1} = 1$ and $g^{-1}g = 1$.

The category \underline{Grp} has object consisting of all groups and arrows (or morphisms as they are also called) all functions $G_1 \xrightarrow{f} G_2$ on the underlying sets that satisfy $f(ab) = f(a)f(b)$ and $f(1) = 1$. It follows that $f(g^{-1}) = f(g)^{-1}$. The proof of this fact is left to the reader.

The set of arrows is denoted as usual by either $[G_1, G_2]_{\underline{Grp}}$ or $hom_{\underline{Grp}}(G_1, G_2)$. Such morphisms are called group maps or homomorphisms (a word that, historically, no doubt inspired the word “morphism” in general).

Remark 5.4 *In all cases of semigroups, monoids, and groups, if the operation satisfies an additional condition called commutativity, viz. $xy = yx$ for all x and y in the underlying set object, the operation is denoted by “+” instead of \cdot (or just juxtaposition) and the identity element is denoted by 0 instead of 1.*

Thus, if G is a commutative group then $x + y = y + x$ for elements of G and for all x in G , we have $x + 0 = x$ and $x + (-x) = 0$. In general, we define $x - y = x + (-y)$ in a commutative group.

Commutative semigroups, monoids, and groups are often called “abelian” in honor of the mathematician Niels Henrik Abel (1802-1829) [web ref](#).

5.10 The Category of Abelian (Commutative) Groups

The category of abelian groups, \underline{AbGprp} consists of all commutative groups with morphisms exactly the same as in \underline{Rgp} .

Remark 5.5 Given a set X with an operation \cdot , we denote the mathematical system consisting of the set with its operation \cdot by (X, \cdot) . If there are any distinguished elements like an identity element 1 , we denote the system by $(X, \cdot, 1)$, etc.

Example 5.3

- The system $(\mathbb{N}, +, 0)$ where $\mathbb{N} = \{0, 1, 2, \dots\}$ is the set of non-negative integers and the operation is the usual one of addition and the identity element is 0 is an abelian monoid.
- The system $(\mathbb{Z}, +, 0, -)$ where $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ and $+$ is the usual operation of addition, 0 is the identity element, and $-$ is the usual operation for negation for inverses is an abelian group.
- The monoid in remark (5.2) consisting of all non-commutative words in 0 and 1 with identity element id_n is a monoid that is not abelian.

Let $\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z} \right\}$ be the set of rational numbers, $\mathbb{R} = \{n.a_0a_1a_2\dots \mid n \in \mathbb{Z}, a_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}\}$ be the set of real numbers, and $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}, i = \sqrt{-1}\}$ be the set of complex numbers.

- The systems $(\mathbb{Q}, +, 0, -)$, $(\mathbb{R}, +, 0, -)$, and $(\mathbb{C}, +, 0, -)$ are all abelian groups with the commutative operation of usual addition, identity element 0 , and $-$ as inverse operation for addition.

5.11 The Category of Commutative Rings with Identity

A commutative ring is a system of the form $(R, +, 0, -, *)$ where $(R, +, 0, -)$ is a commutative (abelian) group and $R \times R \xrightarrow{*} R$ is a commutative (abelian) semigroup such that the following *distributive laws* hold.

$$r * (s + t) = r * s + r * t \quad (57)$$

$$(s + t) * r = s * r + t * r, \text{ for all } r, s, t \in R. \quad (58)$$

As usual, the multiplication operation is often written simply as juxtaposition,

A commutative ring with identity is a system $(R, +, 0, -, *, 1)$ $(R, +, 0, -, *)$ is a commutative ring and $(R, *, 1)$ is a commutative (abelian) monoid.

The category \underline{AbRng}_1 has objects all commutative rings with identity and morphisms all functions $R \xrightarrow{f} S$ where R and S are objects in \underline{AbRng}_1 which are abelian group morphisms with respect to $+$ and abelian monoid morphisms with respect to $*$.

Exercise 5.2 Clearly, the systems $(\mathbb{Z}, +, 0, -, *, 1)$, $(\mathbb{Q}, +, 0, -, *, 1)$, $(\mathbb{R}, +, 0, -, *, 1)$, and $(\mathbb{C}, +, 0, -, *, 1)$ are all commutative rings with identity with respect to the usual operations indicated. Recall the quotient sets \mathbb{Z}/n from exercise (2.2).

Show that $(\mathbb{Z}/n, +, 0, -, *, 1)$ is a commutative ring with identity for every non-negative integer n with the operations given by

$$[a] + [b] = [a + b] \quad (59)$$

$$[a] * [b] = [a * b] \quad (60)$$

$$-a = [-a] \quad (61)$$

and constants $0 = [0]$, $1 = [1]$. Hint, this amounts to showing that the functions defining these operations are well-defined – a simple exercise in arithmetic (when the problem is organized properly).

Remark 5.6 While non-commutative rings are studied in mathematics, only commutative rings with identity will be considered in this particular document.

5.12 The Category of Fields

A field F is a mathematical system such that $(F, +, 0, -, *, 1)$ is a commutative ring with identity and $(F - 0, *, 1, /)$ is a commutative group. Note that the inverse is conventionally written as $r^{-1} = 1/r$ (or $\frac{1}{r}$) and a/b (or $\frac{a}{b}$) is equal to $a * b^{-1}$. Each of \mathbb{Q} , \mathbb{R} , and \mathbb{C} are fields with respect to the usual operation of division. The ring of integers \mathbb{Z} is not a field.

The category \underline{Fld} has objects all fields and arrows all functions that preserve the operations.

5.13 The Category of Modules over a Ring

A module M over a ring R (or an R -module as it is also called) is an abelian group structure $(M, +, 0, -)$ along with an operation (sometimes called an “action”) of R on M , $R \times M \xrightarrow{\mu} M$ that satisfies the properties that, writing $\mu(r, m) = rm$,

$$r(m + m') = rm + rm' \quad (62)$$

$$(r + s)m = rm + sm \quad (63)$$

$$rsm = r(sm) \quad (64)$$

$$1m = m, \text{ if } m \text{ has an identity element.} \quad (65)$$

Remark 5.7 *It is worth noting at this point that if M is an abelian group, then the set of arrows (morphisms) $\text{hom}_{\underline{AbGrp}}(M, M)$ has the structure of a ring with identity element. The operation of addition is given by*

$$(f + g)(m) = f(m) + g(m) \quad (66)$$

and the operation of multiplication is given by

$$fg = f \circ g \quad (67)$$

where \circ denoted composition of functions.

The ring $\text{hom}_{\underline{AbGrp}}(M, M)$ is usually denoted by $\text{End}(M)$ and called the endomorphism ring of M .

Exercise 5.3 *Verify that the operations given above make $\text{End}(M)$ into a ring with identity. Note that, in general, the multiplication is not commutative. Later, this will become clear when we identify $\text{End}(M)$ as a ring of matrices in special cases.*

Furthermore note that if M is an R -module, then $\text{End}(M)$ is also an R -module where $(rf)(m) = rf(m)$ for a module morphism f .

Finally, verify that the function

$$R \xrightarrow{\rho} \text{End}(M) \quad (68)$$

$$r \mapsto (m \mapsto rm) \quad (69)$$

is a ring with identity morphism. The morphism ρ is called a representation of R on M .

The category of modules over a ring is denoted by \underline{RMod} .

5.14 The Category of Vector Spaces Over a Field

For basic information about vector spaces, matrices, and linear algebra in general, the textbooks, [10, 9, 4] are useful.

A vector space is a module V over a field F . The extra operation of division in F considerably enriches the “computational power” of vector spaces compared to modules and this partially accounts for its prevalence as a computational tool in engineering. However do not be misled into thinking that modules are not important computationally as well. The operation of division is quite useful *when it is available*, but that is not always the case.

The category of vector spaces over a field F is denoted by \underline{FVect} so that, of course, $\underline{FVect} = \underline{FMod}$.

Much more will be said about this category shortly.

5.15 The Free R -Module on a Set X

There is an explicit construction of a type of R -module that has useful properties shared with vector spaces. Recall that every vector space V over a field F has a *basis* $B \subseteq V$. Bases are characterized by the fact that every linear combination of the form $\sum_{i=1}^n r_i b_i$ where $r_i \in F$ and $b_i \in B$ is unique. If X is a set and R is a ring, we can construct an R -module M with this “basis property” with respect to $X \subseteq M$, i.e. every linear combination of the form $\sum_{i=1}^n r_i x_i$ where $r_i \in R$ and $x_i \in X$ is unique.

Here is the construction. Let M be the set of all formal linear combinations of the form $\sum_{i=1}^n r_i x_i$ where $r_i \in R$ and $x_i \in X$. To make things completely unambiguous, one can take such a linear combination to mean a

list L of pairs (r, x) where $r \in R$ and $x \in X$. Thus, for example, the linear combination above is represented by the list

$$L = [(r_1, x_1), (r_2, x_2), \dots, (r_n, x_n)] \quad (70)$$

and to be even more explicit, we can take lists to be ordered n -tuples in n -time products of the set $R \times M$. The additive structure on such linear combinations is exactly as it is for vector spaces and the action of R (also called “scalar multiplication”) on M is also as it is for vector spaces. We need the convention that in this context, there is an “empty” list and that is the zero element of M .

We will call this construction the free R -module on X and denote it by $\text{FreeMod}(R, X)$.

6 A Categorical Properties of Modules Over a Ring

Everything done in this section will be true for modules over a ring as well as vector spaces over a field, i.e. no reference or reliance upon the operation of division or more notably on the existence of a basis will be necessary.

Consider the correspondence $\text{Ob}({}_R\text{Mod}) \xrightarrow{D} \text{Ob}({}_R\text{Mod})$ given by

$$D(M) = \text{hom}_{{}_R\text{Mod}}(M, R). \quad (71)$$

This set of morphisms is indeed another R -module since we can add such morphisms by “adding pointwise”, i.e. $(\alpha + \beta)(m) = \alpha(m) + \beta(m)$ and scalar multiplying by $(r\alpha)(m) = r\alpha(m)$ and the necessary relations can easily be checked to see that this turns $D(M)$ into an R -module. $D(M)$ is most often denoted by M^* and is called the dual module of M . It turns out that this correspondence is the first part of a contravariant functor. On home sets (i.e. arrows) the correspondence $D(f) = f^*$ is given as follows. If $V \xrightarrow{f} W$ is a morphism and the morphism $W \xrightarrow{\beta} R$ is given, the corresponding morphism $V \longrightarrow R$ is given by $f^*(\beta)(v) = (\beta \circ f)(v) = \beta(f(v))$.

Exercise 6.1 *Show that D is indeed a contravariant functor by proving that it preserves composition of morphisms. It would be helpful to write out diagrams for all of this.*

Exercise 6.2 *Show that the composition of two functors is a functor.*

Much more needs to be said about the categorical properties of R -modules and vector spaces over a field. This will have to be left to an extension of these notes that is in preparation.

7 Cocones and a Look Toward Future Developments

A diagram in a category \underline{C} is the image of a functor $D \xrightarrow{F} \underline{C}$. A cocone in \underline{C} over F with vertex $C \in \underline{C}$ is a correspondence collection of morphisms $F(z) \xrightarrow{c_x} C$ such that all diagrams of the form

$$\begin{array}{ccc} F(x) & \xrightarrow{\xi} & F(y) \\ & \searrow c_x & \swarrow c_y \\ & C & \end{array} \quad (72)$$

where $\xi = F(f)$ and $f \in \text{hom}_{\underline{D}}(x, y)$ commute, i.e. $c_x \xi = c_y$.

The vertex of a cocone as above is said to be a colimit if every morphism from an object C to another object C' in \underline{C} which is also a vertex of a cocone over F is *uniquely* determined by morphisms from the $F(z)$ to C' for which the maps within the cocone for C' are “compatible” with the morphisms in the cocone for C in a sense that will be made precise in the example below.

7.1 Coequalizers

A coequalizer is a kind of colimit. The corresponding cocone involves the category generated by the digraph (D_0, D_1, s, t, ids) where $G_0 = \{x, y\}$, $G_1 = \{id_x, id_y, \alpha, \beta\}$, $s(a) = x$ and $t(a) = y$ for $a \in G_1$. Thus, G has the shape

$$x \begin{array}{c} \xrightarrow{\alpha} \\ \xrightarrow{\beta} \end{array} y \quad (73)$$

Since there are no path products other than the compositions with the identities, the entire category \underline{D} generated by G has the same shape. So a diagram in \underline{C} over a functor F is of the form

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B \quad (74)$$

where $F(x) = A$, $F(y) = B$, $F(\alpha) = f$, and $F(\beta) = g$.

A cocone with vertex C then consists of a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\xi} & B \\ & \searrow c_x & \swarrow c_y \\ & C & \end{array} \quad (75)$$

where $\xi \in \{f, g\}$. Now note that given such a situation, we may define a diagram

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B \xrightarrow{c} C \quad (76)$$

where $c = c_y$ and then we have $cf = cg$ from the cocone condition. Conversely however, if we have a diagram such as the one above, we may form a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\xi} & B \\ & \searrow c_x & \swarrow c_y \\ & C & \end{array} \quad (77)$$

where $\xi \in \{f, g\}$ and $c_x = cf$ and $c_y = c$. Thus the two situations are equivalent. We call the object C in the situation in equation (76) a coequalizer if for any any other diagram of the form

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B \xrightarrow{c'} C' \quad (78)$$

where c' with $c'f = c'g$ there is a unique morphism $C \xrightarrow{\phi} C'$ such that $c' = \phi \circ c$. Note that this means that any morphism from C to another object C' is completely determined by a map c' from B to C' for which $c'f = c'g$.

Exercise 7.1 *Translate these last statements back to the cocone situation and realize that essentially it is saying that any map from a colimit object is given uniquely by a map from the “parts” of the cocone of which it is the vertex as long as some compatibility conditions are satisfied.*

Colimits in Set and many other categories can be constructed using coequalizers. Obviously, for this to be useful, one needs to know how to construct coequalizers in such categories. The basic idea is to take $C = B/E$

where E is the equivalence relation generated by the relation $f(a) \xrightarrow{E} g(a)$ for all $a \in A$. Turning this into a finite effective algorithm can be done, but it takes much more discussion than we have available at this time to understand that. See the comments below.

8 Conclusions

From discussions with the Consortium, it does appear that category theory can be a useful tool for organizing ideas in systems engineering. Much more foundation for what is needed must be developed beyond what we have presented here. The only constraint to that happening is time and proper funding for this research.

Discussions with Michael Watson have indicated that categories in which nodes and arrows are defined by time dependent stochastic processes are not only feasible, but have the potential of being quite useful. Much more in the direction of uncertainty quantification of categorical models for complex systems needs to be developed, and the author of this note, based on conversations with members of the Consortium is convinced that such research will be fruitful and useful in engineering in general.

8.1 A Working Definition

A working definition of “system” was given in the webinar SE Webinar. A print version of the presentation is available at print version. In the Webinar, a definition of system as a recursive colimit was given. That working definition has been revised and can be found in sys.def. The definition is repeated here.

Definition 8.1 *A cocone is atomic if all the nodes comprising it except the vertex are considered to be indecomposable, i.e. not vertexes of cocones themselves.*

A system is the vertex of either an atomic final cocone or a final cocone whose nodes are recursively systems.

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